On the reducibility of the discrete linear time-varying systems

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Abstract. For the discrete linear time-varying systems we present basic facts and definitions concerning the Lyapunov transformation, kinematic similarity and reducibility in the context of stability and Lyapunov exponents theory. Moreover, the paper contains the original result giving the necessary and sufficient conditions for the reducibility of a system to system with identity matrix.

Introduction

In the literature, many publications and books are devoted to the linear systems with constant coefficients, partially because of the fact, that their solutions can be obtained in the explicit form. A wider class of systems are systems with periodic coefficients, which solutions can also be obtained in the above-mentioned way. Lyapunov in the paper [1] introduced the notion of the transformation called today the Lyapunov transformation, which does not change the character of solutions dynamic. The systems, for which there exists the Lyapunov transformation, that transform from one of them into the second are called kinematically similar. Such a relation is an equivalence relation. The description of equivalence classes with respect to this relation is an important issue in studying dynamical properties of the discrete linear time-varying systems. The paper [2] is devoted to this problem. In the literature [3] the systems, which by using Lyapunov transformation can be transformed to the systems with constant coefficients, are called reducible systems. In the paper we consider the problem of kinematic similarity, in particular the reducibility of the discrete linear time-varying systems.

Definitions and basic properties

Consider the discrete time-varying system of the form:

$$x(n+1) = A(n)x(n), \quad n \geq 0, \tag{1}$$

where $A = \left(A(n)\right)_{n\in\mathbb{N}}$ is a bounded sequence of invertible s-by-s real matrices such that $\left(A^{-1}(n)\right)_{n\in\mathbb{N}}$ is bounded. By $\|\cdot\|$ let us denote the Euclidean norm in $\mathbb{R}^s$ and the induced operator norm, and by $\mathbb{R}^{s\times s}$ the set of all s-by-s real matrices. Moreover $\prod_{i=0}^{n-1} A(i) = A(n-1)\ldots A(0)$. For the system (1) a transition matrix $\Phi(n,m)$ is equal to $A(n-1)\ldots A(m)$ for $n > m$ and

$\Phi(n,n) = I$, where I is the identity matrix. For an initial condition $x_0 \in \mathbb{R}^s$ the solution of (1) is denoted by $x(n,x_0)$, so $x(n,x_0) = \Phi(n,0)x_0$. If $x_0^{(1)}, \dots, x_0^{(s)}$ is the base of $\mathbb{R}^s$ then the matrix $X(n) = \left[x\left(n, x_0^{(1)}\right), x\left(n, x_0^{(2)}\right), \dots, x\left(n, x_0^{(s)}\right)\right]$ we will call fundamental matrix of the system (1). The relation between the transition matrix and fundamental matrix is as follows $X(n) = \Phi(n,0)C$, where C is the invertible matrix defined as $C = \left[x_0^{(1)}, \dots, x_0^{(s)}\right]$. From the control theory point of view, one of the main dynamical systems property is their stability. For discrete linear time-varying systems there exist many not equivalent stability concepts (for example asymptotic, exponential, uniform) [4]. With considered in our paper the Lyapunov transformation and exponents exponential stability is connected, which in turn can be formulate on a few alternative, but equivalent, ways as the Theorem shows.

Theorem 1 *For system (1) the following 3 conditions are equivalent*

$$\bigvee_{0\le\lambda<1} \bigwedge_{x_0\in\mathbb{R}^s} \bigvee_{\mu(x_0)\ge1} \bigwedge_{n\ge0} \|\Phi(n,0)x_0\| \le \mu(x_0)\lambda^n \tag{2}$$

$$\bigvee_{0\le\lambda<1} \bigvee_{\mu\ge1} \bigwedge_{n\ge0} \|\Phi(n,0)\| \le \mu\lambda^n \tag{3}$$

$$\bigvee_{0\le\lambda<1} \bigvee_{\mu\ge1} \bigwedge_{n\ge0} \bigwedge_{x_0\in\mathbb{R}^s} \|\Phi(n,0)x_0\| \le \mu\lambda^n\|x_0\| \tag{4}$$

The proof of the Theorem 1 may be found in [4].

Definition 1 *If any condition (so each one) of the Theorem 1 is satisfied, then the system (1) will be called exponentially stable.*

One of the most important tools using to describe the exponential stability [5,6] of the systems (1) are numerical characteristics called the Lyapunov exponents [7,8,9,10,11,12,13,14,15].

Definition 2 *For* $x_0 \in \mathbb{R}^s$ *the Lyapunov exponent* $\lambda_A(x_0)$ *of (1) is defined as*

$$\lambda_A(x_0) = \limsup_{n\to\infty} (\|x(n,x_0)\|)^{\frac{1}{n}} \tag{5}$$

It is well known [16] that, if A is bounded, then the set of all Lyapunov exponents of system (1) contains at most s elements. The Lyapunov exponents of the system (1) play in the stability theory of discrete linear time-varying systems the similar role as modules of eigenvalues system matrices in stationary systems theory. A generalization of the Lyapunov exponents to the case with unbounded coefficients is proposed in [17,18].

Theorem 2 *[15] The system (1) is exponentially stable if and only if* $\bigwedge_{x_0\in\mathbb{R}^s} \lambda_A(x_0) < 1$.

In practice, we often do not know the precise value of the system coefficients, and therefore we have to estimate the value of the Lyapunov exponents. This problem is studied in [19]. Now, let us define the Lyapunov sequence and equivalent systems.

Definition 3 *The sequence* $L = \left(L(n)\right)_{n\in\mathbb{N}}$ *of* s*-by-*s *real matrices will be called the Lyapunov sequence if it is a bounded sequence, all matrices* $L(n)$ *are invertible and the sequence* $L^{-1} = \left(L^{-1}(n)\right)_{n\in\mathbb{N}}$ *is a bounded sequence.*

If in the equation (1) we change variables according to the formula $x(n) = L(n)y(n)$, then we obtain

$$y(n+1) = B(n)y(n), \tag{6}$$

where

$$B(n) = L^{-1}(n+1)A(n)L(n). \tag{7}$$

Definition 4 *We say that the systems (1) and (6) are kinematically similar, if there exist the Lyapunov sequence L such that the equality (7) holds. If the system (1) is kinematically similar to the system (6) with constant sequence $B = \left(B(n)\right)_{n\in\mathbb{N}}$, then we will say that it is reducible.*

It turns out that the relation of kinematic similarity is reflexive, symmetric and transitive [2], so it is the equivalence relation. The important characteristics of the reducible continuous-time systems was showed in [3], while the following discrete analog may be found in [20].

Theorem 3 *The system (1) is reducible if and only if certain its fundamental matrix has the following form*

$$X(n) = L(n)D^n, \tag{8}$$

where L - a Lyapunov sequence, D – a constant invertible matrix.

In the paper [20], the above-mentioned theorem is presented without the assumption of boundedness of the matrix D, and in this form is not true what we can be shown on a simple example $D = 0$. It should be noted, that if a fundamental matrix of the system (1) has the form defined by (8), then every matrix has such a form. Indeed, let us consider any fundamental matrix $X_1(n)$ of the system (1). Then $X_1(n) = X(n)C$ for certain invertible matrix C. Therefore

$$X_1(n) = L(n)B^nC = L(n)CC^{-1}B^nC = L(n)C(C^{-1}BC)^n. \tag{9}$$

Moreover, if $L(n)$ is the Lyapunov sequence, so it is $L(n)C$.

Consider now the system (1) with periodic sequence A with period K. To characterize the reducibility of discrete linear periodic systems we will use the Floquet Theorem [21].

Theorem 4 *Every fundamental matrix of the discrete linear periodic system with period K has the following form*

$$X(n) = F(n)D^n, \tag{10}$$

where $F(n)$ is a periodic sequence with period K of invertible matrices and D is a constant invertible matrix.

It should be stressed out, as it was proved in [21], that formula (10) is also valid with weaker assumption about invertibility of the matrices $A(n)$, namely with the assumption that the rank of the matrix $A(j+i-1)\dots A(j+1)A(j)$ is for every $i = 1, \dots, s$ constant and independent of $j = 1, \dots, K$.

From the Theorems 3 and 4 we obtain the next conclusion.

Conclusion 1 *Every discrete linear periodic system with invertible coefficients is reducible.*

Main result

Let us start this section with the following theorem.

Theorem 5 *The kinematically similar systems have the same Lyapunov exponents.*

Proof *If the systems (1) and (6) are kinematically similar, then*

$$\bigwedge_{x_0\in\mathbb{R}^s}\bigvee_{y_0\in\mathbb{R}^s} x(n,x_0)=L(n)y(n,y_0) \tag{11}$$

Therefore

$$\lambda_A(x_0)=\limsup_{n\to\infty}\|x(n,x_0)\|^{\frac{1}{n}}=\limsup_{n\to\infty}\|L(n)y(n,y_0)\|^{\frac{1}{n}}\le\limsup_{n\to\infty}\|L(n)\|^{\frac{1}{n}}\|y(n,y_0)\|^{\frac{1}{n}}=\limsup_{n\to\infty}\|y(n,y_0)\|^{\frac{1}{n}}=\lambda_B(y_0). \tag{12}$$

The last equality is the result of the boundedness of $L(n)$*. Now*

$$\lambda_B(y_0)=\limsup_{n\to\infty}\|y(n,y_0)\|^{\frac{1}{n}}=\limsup_{n\to\infty}\|L^{-1}(n)x(n,x_0)\|^{\frac{1}{n}}\le\limsup_{n\to\infty}\|L^{-1}(n)\|^{\frac{1}{n}}\|x(n,x_0)\|^{\frac{1}{n}}=\limsup_{n\to\infty}\|x(n,x_0)\|^{\frac{1}{n}}=\lambda_A(x_0). \tag{13}$$

The last equality is the result of the boundedness of $L^{-1}(n)$*. Finally* $\lambda_A(x_0)=\lambda_B(y_0)$ *what ends the proof of the theorem.*

Among all discrete linear reducible systems on especially comment merit these ones, which can be reduce to the stationary system with identity matrix. The next theorem is taken from [20].

Theorem 6 *Suppose that all solutions of (1) are bounded and* $\prod_{i=0}^{n-1}|detA(n)|\ge a$ *for certain positive* a*. Then the system (1) is kinematically similar to the system*

$$y(n+1)=y(n). \tag{14}$$

The main result of our paper is the content of the next alternative condition on the reducibility of system (1) to the stationary one with identity matrix.

Theorem 7 *If the products* $\prod_{n=0}^{\infty}\|A(n)\|$ *and* $\prod_{n=0}^{\infty}\|A^{-1}(n)\|$ *are convergent, then the system (1) is kinematically similar to the system (14).*

In the proof of the above theorem we will use the next lemma.

Lemma 1 *For any norm* $\|\cdot\|_{**}$ *there exists a constant* c *such that for every invertible matrix* $A\in\mathbb{R}^{s\times s}$ *the following inequality* $|detA|\ge\left(\frac{c}{\|A^{-1}\|}\right)^s$ *holds.*

Proof of the Lemma 1 *Consider* $B\in\mathbb{R}^{s\times s}$*. Denote the columns of the matrix* B *in the following way* $B=[b_1 \ \ ... \ \ b_s]$*. From the Hadamard inequality [22] we have* $|detB|\le\prod_{i=1}^{s}\|b_i\|$*. Using the inequality of arithmetic and geometric means for* $\|b_i\|$*,* $i=1,\dots,s$ *we obtain* $|detB|\le\frac{\left(\sum_{i=1}^{s}\|b_i\|\right)^s}{s^s}$ *and* $|detB|\le\frac{\|B\|_*^s}{s^s}$*. The last inequality is valid for any matrix* B*. Using this inequality for* $B=A^{-1}$

we obtain $\|A^{-1}\|_*^{-s} \leq \frac{|detA|}{s^s}$. *From the matrix norm equivalence [22] arises the fact, that for any norm* $\|\cdot\|_{**}$

$$\bigvee_{\alpha,\beta\in\mathbb{R}} \bigwedge_{B\in\mathbb{R}^{s\times s}} \beta\|B\|_{**} \leq \|B\|_* \leq \alpha\|B\|_{**} \tag{15}$$

therefore $\left(\frac{\frac{s}{a}}{\|A^{-1}\|_{**}}\right)^s \leq |detA|$ *what ends the proof of the lemma.*

Proof of the Theorem 7 *We will show that the assumptions of Theorem 6 are satisfied. We have* $\|x(n,x_0)\| = \|\Phi(n,0)x_0\| \leq \|A(n-1)\| \ldots \|A(0)\|\|x_0\|$. *Because the product* $\prod_{n=0}^{\infty}\|A(n)\|$ *is convergent, then the sequence* $\left(\prod_{n=0}^{m}\|A(n)\|\right)_{m\in\mathbb{N}}$ *is bounded as well. It implies, that* $x(n,x_0)$ *is bounded. According to the Lemma 1 used to the matrix* $A = \prod_{i=0}^{n-1} A(i)$ *we have*

$$det \prod_{i=0}^{n-1} A(i) \geq \left(\frac{c}{\left\|\left(\prod_{i=0}^{n-1} A(i)\right)^{-1}\right\|}\right)^s \tag{16}$$

and therefore

$$\left|det \prod_{i=0}^{n-1} A(i)\right| \geq \left(\frac{c}{\prod_{i=0}^{n-1}\|A^{-1}(i)\|}\right)^s. \tag{17}$$

Because the product $\prod_{i=0}^{\infty}\|A^{-1}(i)\|$ *is convergent, then the sequence* $\prod_{i=0}^{n-1}\|A^{-1}(i)\|$ *is bounded. Denote by* a *its bound. Then, from the inequality (17) we obtain*

$$\left|det \prod_{i=0}^{n-1}\|A(i)\|\right| \geq \left(\frac{c}{d}\right)^s > 0. \tag{18}$$

Conclusions

In the paper we have investigated the properties of kinematic similarity and reducibility of discrete linear time-varying systems in the context of stability and Lyapunov exponents theory. Our main result, contained in the Theorem 7, has shown new sufficient condition for reducibility to a system with identity matrix.

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