Trajectory controllability of semilinear systems with delay in control and state

Jerzy Klamka[1, a], Elżbieta Ferenstein[2, b], Artur Babiarz[1, c], Adam Czornik[1, d]

and Michał Niezabitowski[1, e]

[1] Silesian University of Technology, Institute of Automatic Control,

16 Akademicka Street, 44-101 Gliwice, Poland

[2] Polish-Japanese Institute of Information Technology,

Aleja Legionów 2, 41-902 Bytom, Poland

[a] Jerzy.Klamka@polsl.pl , [b] elaw@pjwstk.edu.pl , [c] Artur.Babiarz@polsl.pl ,

[d] Adam.Czornik@polsl.pl , [e] Michal.Niezabitowski@polsl.pl

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Abstract. In this paper we consider the finite-dimensional dynamical control system described by scalar semilinear ordinary differential state equation with variable delay. The semilinear state equation contains both pure linear part and nonlinear perturbation. We extend the concept of controllability on trajectory controllability for systems with point delay in control and in nonlinear term. Moreover, we present remarks and comments on the relationships between different concepts of controllability. Finally we propose the possible extensions.

Introduction

In mathematical control theory there are a few important concepts. One of them is controllability [1,2,3], which in general means, that it is possible to steer dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Wide research into controllability was started in the 1960s. The concept of controllability was first presented by Kalman and refers to linear dynamical systems. Because the most of practical dynamical systems are nonlinear, that's why, in recent years various controllability problems for different types of nonlinear or semilinear dynamical systems have been considered in many publications and monographs. Simultaneously there are more and more of tools used to investigate into controllability problems, for example, fixed point theorems and measures of noncompactness for function spaces [4]. Also, in the literature there are many different definitions of controllability, namely complete controllability [5], approximate controllability [1,2], exact controllability [1,2], trajectory-controllability [8].

In the paper we study the trajectory controllability of semilinear finite-dimensional first-order dynamical systems with variable delay in control and in nonlinear term. In general, the trajectory controllability means, that it is possible to steer the dynamical control system from an arbitrary initial state to an arbitrary final state, along a prescribed trajectory, using the set of admissible controls. Moreover, for simplicity of considerations, it is generally assumed that the values of admissible controls are unconstrained. In effect, T-controllability is a stronger notion than controllability. The motivation to study the T-controllability implies from the fact, that most of industrial processes are nonlinear in nature.

The main object of the paper is to extend the results given in papers ([9,10]) to the semilinear dynamical system with variable delay in control and in nonlinear term.

System description

Consider the semilinear nonstationary finite-dimensional control system described by the following scalar first-order ordinary differential equation with variable delay in control and state variables

$$\dot{x}(t) = a(t)x(t) + b(t)\,u\big(t - h(t)\big) + f\left(t, x(t), x\big(t - h(t)\big)\right) \quad (1)$$

defined for $t \in [t_0, T]$, $t > t_0$, $h(t)$ is given differentiable variable delay. Since, $t - h(t) > 0$ is increasing function of time t than $\frac{d}{dt}\big(t - h(t)\big) > 0$ and thus $\big(1 - \dot{h}(t)\big) > 0$, $\dot{h}(t) < 1$ and $h(T) \leq T$. Moreover, $x(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}$ are scalar state variable and scalar admissible unbounded control, respectively and $u \in L^2[t_0, T]$.

For simplicity of considerations we assume zero initial conditions both for the state variable and control. Therefore, we have $x(t) = 0$ and $u(t) = 0$ for $t \in [t_0 - h(t_0), t_0]$. Let us also define functions:

$$v(t) = t - h(t) \quad (2)$$

for $t \in [t_0, T)$. Since $h(t)$ is differentiable than the function

$$v : [t_0, T) \to \mathbb{R} \quad (3)$$

is also differentiable and strongly increasing $\big(\dot{v(t)} > 0$ in $[t_0, T]\big)$.

Moreover, let us suppose that $a(t)$, $b(t)$ are given scalar continuous functions defined on $t \in [t_0, T]$. As well, it is assumed that

$$f : ([t_0, T]) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad (4)$$

is given nonlinear function and measurable with respect to the first variable and continuous with respect to the second and the third variable. Furthermore, it should be notice that there is no constraints imposed on the state variable $x(t)$ and the admissible control values $u(t)$. For simplicity of considerations we introduce an overtaking function

$$r : [v(t_0), v(T)] \to [t_0, T], \quad (5)$$

which is the inverse function to $v(t)$, hence $r(t) = t + h(t)$ for $t \in [t_0, T)$.

Next, let us recall the well-known lemma [1] concerning relationships between the ordinary differential state equations with delay in control and the ordinary differential equations without delay in control. The next lemma will be useful in our further consideration.

Lemma 1 *Dynamical system (1) can be expressed as follows:*

$$\dot{x}(t) = a(t)\,x(t) + b_T(t)\,u(t) + f\left(t, x(t), x\big(t - h(t)\big)\right) \quad (6)$$

for $t \in [t_0, T]$, *where*

$$b_T(t) = F\big(t_0, r(t)\big)\,b\big(r(t)\big)\,\dot{r}(t) \quad (7)$$

for $t \in [t_0, v(T))$ *and*

$$F\big(t_0, r(t)\big) = e^{\int_{t_0}^{r(t)} a(\tau)d\tau}. \quad (8)$$

Remark 1 *Let us observe, that function* $b_T(t)$ *strongly depends on time* T.

Now, for completeness of considerations, let us recall some fundamental definitions of controllability. It's well-known [1] that for the finite-dimensional first-order dynamical system (6) it is possible to define many different concepts of controllability.

Definition 1 *Dynamical system (6) is said to be completely controllabile on* $t \in [t_0, T]$ *if for any* $x_0, x_1 \in \mathbb{R}$ *there exist a control* $u \in L^2([t_0, T])$ *such that the corresponding solution* $x(\cdot)$ *of (6) satisfies* $x(T) = x_1$.

Remark 2 *It should be pointed out, that according to the Definition 1 there is no constraints imposed on trajectory between* t_0 *and* T. *Moreover, let us observe that we don't know anything about trajectory along which the system moves.*

From the real industrial processes point of view, it would be necessary to steer dynamical process from a given initial state to final state along the prescribed trajectory. Practically, it would be desirable to choose more suitable path which reduce cost of steering process. Hence, in the literature [10] appears a new, stronger notion of controllability, namely the Trajectory controllability (in short T-controllability).

Let τ be a given set of all differentiable functions $z(\cdot)$ defined on $t \in [t_0, T]$, which satisfy the initial and final conditions $z(t_0) = 0$, $z(T) = x_1$ and $z(\cdot)$ is a prescribed trajectory differentiable almost everywhere in $[t_0, T]$. Then, we have the following definition of T-controllability.

Definition 2 *[10]. Dynamical system (6) is said to be T-controllable if for any* $z \in \tau$, *there exists a control* $u \in L^2([t_0, T])$ *such that the corresponding solution* $x(\cdot)$ *of (6) satisfies* $x(t) = z(t)$ *in* $[t_0, T]$.

From the Definitions 1 and 2 immediately follows the following corollary.

Corollary 1 *Suppose that system (6) is T-controllable in time interval* $[t_0, T]$. *Then a dynamical system (6) is completely controllable in time interval* $[t_0, T]$.

T-controllability of semilinear dynamical systems with variable delay

In this section we study T-controllability in a given time interval $[t_0, T]$ for semilinear scalar dynamical system variable delay (1) using the associated differential equation (6) of dynamical system without delay in control. The main result of the paper is the following sufficient condition for T-controllability:

Theorem 1 *Suppose that:*

1. *the functions* $a(t)$ *and* $b(t)$ *are continuous on* $t \in [t_0, T]$;
2. $b_T(t)$ *do not vanish on* $t \in [t_0, T]$;
3. *Function* $f(\cdot,\cdot,\cdot)$ *is Lipschitz continuous with respect to the second and third arguments, i.e. there exist positive constants* α_1 *and* α_2 *such as* $|f(t,x,z) - f(t,y,z)| \le \alpha_1 |x - y|$ and $|f(t,x,z) - f(t,y,w)| \le \alpha_2 |z - w|$.

Then, the first-order semilinear dynamical system with variable delay described by state equation (1) is T-controllable in a given time interval $[t_0, T]$, *with respect to every differentiable trajectory* $z(\cdot) \in \tau$.

Proof *From the Lipschitz continuity of the function* $f(\cdot,\cdot,\cdot)$ *implies that for every given admissible control* $u(t)$ *there exists a unique solution* $x(t)$ *for the system (6). Let* $z(t)$ *be a given differentiable trajectory in the set* τ. *We extract admissible control function* $u(t)$ *from differential equation (6). Hence, we have*

$$u(t) = \frac{\dot{z}(t) - a(t)z(t) - f\big(t, z(t), z(t - h(t))\big)}{b_T(t)}. \tag{9}$$

Moreover, with admissible control function (9), the state equation (6) becomes:

$$\dot{x}(t)-\dot{z}(t)=a(t)\big(x(t)-z(t)\big)+f\Big(t,x(t),x\big(t-h(t)\big)\Big)-f\Big(t,z(t),z\big(t-h(t)\big)\Big). \quad (10)$$

Let us make the substitution $e(t)=x(t)-z(t)$. *Then, for* $e(t)$ *we have ordinary differential equation of the following form*

$$\dot{e}(t)=a(t)e(t)+f\Big(t,x(t),x\big(t-h(t)\big)\Big)-f\Big(t,z(t),z\big(t-h(t)\big)\Big). \quad (11)$$

From the theory of ordinary differential equations it follows, that the solution of equation (11) with zero initial condition $e(t_0)=0$ *is described as*

$$e(t)=\int_{t_0}^{T}\Phi(t,s)f\Big(s,x(s),x\big(s-h(s)\big)\Big)-\int_{t_0}^{T}\Phi(t,s)\,f\Big(s,z(s),z\big(s-h(s)\big)\Big)\,ds, \quad (12)$$

where fundamental solution of linear part of equation is in the following form $\phi(t,s)=e^{\int_s^t a(s)ds}$. *Therefore, from (12) we obtain*

$$e(t)\le\int_{t_0}^{T}|\Phi(t,s)|\alpha_1|x(s)-z(s)|ds+\int_{t_0}^{T}|\Phi(t,s)|\alpha_2\big|x\big(s-h(s)\big)-z\big(s-h(s)\big)\big|ds \quad (13)$$

Hence,

$$|x(t)-z(t)|\le\alpha_1\int_{t_0}^{T}|\Phi(t,s)||x(s)-z(s)|ds+\alpha_2\int_{t_0}^{T}|\Phi(t,s)|\big|x\big(s-h(s)\big)-z\big(s-h(s)\big)\big|ds.$$
(14)

Using the Gronwall's inequality [8,11] it follows that

$$\|x(t)-z(t)\|_{L^2[t_0,T]}=0 \quad . \quad (15)$$

However, since $x(t)$ *and* $z(t)$ *are in fact continuous functions, then* $x(t)=z(t)$ *for* $t\in[t_0,T]$. *This proves T-controllability of the dynamical system (1) in the time interval* $[t_0,T]$.

Example

Now, let us consider T-controllability of the simple illustrative example. Let the first-order finite-dimensional dynamical control system with variable delay in control defined on a given time interval $[t_0,T]$, has the following form:

$$\dot{x}(t)=a(t)x(t)+b(t)u\big(t-h(t)\big)+cosx\big(t-h(t)\big) \quad (16)$$

for $t\in[t_0,T]$, $T>h$, where $t_0=1$, $T=2$ and $h(t)=-t^2+1$. Therefore, for $t\in[1,2]$ we obtain $1-h(t)=t^2+t-1>0$ and $1-\dot{h}(t)=-2t<0$. Moreover

$$f\big(t,x(t),x(t-h)\big)=cosx\big(t-h(t)\big) \quad (17)$$

and functions $a(t)$ and $b(t)$ are continuous in $[t_0,T]$. Also, $f\big(x(t-h)\big)$ satisfies Lipschitz condition. Hence, all assumptions of the Corollary 2 are satisfied and therefore, semilinear scalar dynamical system (16) is T-controllable on $[t_0,T]$.

Conclusion

In the paper sufficient conditions for T-controllability of semilinear differential equation with variable point delay have been formulated and proved. These conditions are the extension to the case of T-controllability of first-order dynamical control systems without delays [8]. Finally, it should be pointed out, that the obtained results could be extended in many directions: discrete-time [10,12,13,14]; second-order dynamical control [6,15]; switched systems [16,17,18,19,20,21,22,23]; linear fractional discrete-time [24,25] and infinite-dimensional systems [26]. Moreover, controllability is strongly connected with the so-called minimum energy control problem [27,28,29,30,31,32,33,34,35].

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